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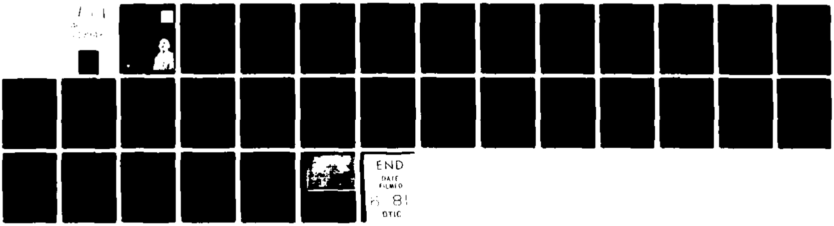

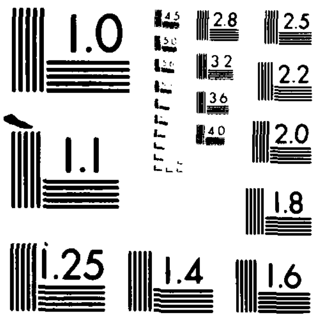
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FOURIER INTEGRAL ESTIMATE OF THE FAILURE RATE FUNCTION AND ITS MEAN SQUARE ERROR PROPERTIES

by

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Man-Yuen Wong

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20. Abstract (continued)

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1. Introduction and Summary

The failure rate function h is an important parameter in reliability and biometry. Estimates of h using weighting functions or "kernels" are quite common in the literature [see Rice and Rosenblatt (1975) and related references]. The kernels that have been considered so far are *nonnegative* and *absolutely integrable* in $(-\infty,\infty)$. (Kernels satisfying this latter condition are known as L^1 kernels.) In Singpurwalla and Wong (1980)--abbreviated as SW(1980)--we have shown that the mean square error (MSE) of a kernel estimator of h using an L^1 kernel restricted to be nonnegative has a rate of convergence of at most $O(n^{-4/5})$,* regardless of the smoothness of h; where n is the sample size. If the nonnegativity condition of an L^1 kernel is relaxed, and if h is m times continuously differentiable, then (for $m > 2$), the rate of convergence of the MSE (can be improved and) is at

*The notation "$a_n = O(b_n)$" denotes the fact that $|a_n/b_n|$ is bounded in the limit.

most $O(n^{-\frac{2m}{2m+1}})$. A method for producing kernel estimators having the above property is the generalized jackknife of Gray and Schucany (1972). Specifically, if we use the generalized jackknife on two kernel estimators of h, with each estimator being based upon a nonnegative L^1 kernel, then this is equivalent to directly producing a kernel estimator of h using an L^1 kernel which takes both positive and negative values. If we continue to apply the generalized jackknife method, then the rate of convergence of the MSE of the resulting estimator can be brought as close to n^{-1} as is desired. This, plus the alternating behavior of the kernel discussed in Example 5.3 of SW(1980), has prompted us to conjecture that an indefinite jackknifing of estimators based on L^1 kernels is equivalent to obtaining a kernel estimator using an alternating (wave-like) non L^1 kernel.

Motivated by the above considerations, out goal in this paper is to obtain an estimator of h whose MSE converges to 0 faster than $O(n^{-\frac{2m}{2m+1}})$ for any finite $m > 0$, and preferably is closer to the ideal n^{-1}. We achieve this goal by considering a kernel estimator of h based on the "sinc" kernel, illustrated in Figure 3.1. In Section 3 we show that the sinc kernel, which is not an L^1 kernel and may not be a limiting case of jackknifing an L^1 kernel either, arises naturally when we estimate h via an estimate of the Fourier transform of h. The sinc kernel estimator of h is also referred to as the "Fourier integral estimate."

In Section 4 we show that the sinc kernel estimators of h are asymptotically unbiased and consistent. In Section 5, we discuss the rates of convergence of the bias and the MSE of these estimators. We show that for certain classes of failure rate functions, the sinc kernel estimators have a faster rate of convergence of the MSE than the

corresponding L^1 kernel estimators. These rates are of the order $(\log n/n)$ or $(n^{\frac{1}{2p-1}-1})$, depending upon whether the Fourier transform of h decreases "exponentially" or "algebraically with degree p" (see Definitions 5.1 and 5.2). Clearly, when $p > m+1$, both the above rates are faster than $n^{-\frac{2m}{2m+1}}$. However, they are not equal to the ideal rate of n^{-1}, which we know can be attained, if possible, by jackknifing indefinitely.

Sinc kernels have been considered before in the literature on density estimation. Davis (1975) has shown that under certain conditions density estimates based on the sinc kernels have a *faster rate of* convergence of the MSE than those based on L^1 kernels. Thus, the results of our paper complement those of Davis. An explanation of why the sinc kernel gives us faster rates of convergence of the MSE lies in the effect of jackknifing on kernels. Typically, the generalized jackknife method is undertaken to improve upon the rate of convergence of the bias and the MSE, and an indefinite jackknifing using L^1 kernels leads us to a non L^1 kernel, which alternates between positive and negative values, like the sinc function.

2. Preliminaries: Kernel Estimates

Suppose that the time to failure of a device is a nonnegative random variable X, with an absolutely continuous distribution function F and a probability density function f. The *failure rate* at x_0, $h(x_0)$, for $F(x_0) \neq 1$, is defined as

$$h(x_0) = \frac{f(x_0)}{1 - F(x_0)} ;$$

note that $h(x) \geq 0$, for all $x \geq 0$.

Given an ordered sample of n lifetimes from F, say $X_{(1)}$, ..., $X_{(n)}$, a *kernel estimate* of $h(x_0)$, $h(n,x_0)$, is defined as

$$h(n,x_0) = \sum_{j=1}^{n} \frac{1}{n-j+1} \frac{1}{b(n)} K\left(\frac{X_{(j)} - x_0}{b(n)}\right), \tag{2.1}$$

where the kernel K is a bounded, symmetric function of integral one; the scale parameter $b(n)$ is a nonnegative decreasing function of n such that

$$\begin{aligned} &\text{(i)} \quad \lim_{n\to\infty} b(n) = 0, \quad \text{and} \\ &\text{(ii)} \quad \lim_{n\to\infty} n\, b(n) = \infty . \end{aligned} \tag{2.2}$$

A motivation for considering the kernel estimates of the failure rate are given in Watson and Leadbetter (1964a). Some of the commonly used kernels are the rectangular, the triangular, the Weierstrass, the Piccard, the Cauchy, and the kernel $K(x) = \frac{1}{2\pi}\left(\frac{\sin(x/2)}{(x/2)}\right)^2$. Among other things, these kernels have the following two features which are of interest:

(i) they are nonnegative, and

(ii) they are absolutely integrable in $(-\infty,\infty)$; kernels which satisfy this property are called L^1 kernels.

Watson and Leadbetter (1964b) have shown that for a certain class of distribution functions, estimates based on L^1 kernels are asymptotically unbiased and consistent, at every point x at which h is continuous and $F(x) < 1$. The optimal rates of convergence of the bias and the MSE of $h(n,x_0)$ have been discussed by SW(1980).

3. Kernel Estimates Based on the Fourier Integral

We shall confine our attention to the class of failure rate functions h for which the Fourier transform Φ_h exists; that is

$$\Phi_h(x) = \int e^{ixu}\, h(u)du < \infty .$$

Let x_0 be a point of continuity of $h(x)$, and assuming that $\Phi_h \varepsilon L^1$ (i.e., $\int|\Phi_h(x)|dx < \infty$) , the following inversion formula gives us the basis for considering the Fourier integral estimate of the failure rate:

$$h(x_0) = \frac{1}{2\pi} \int e^{-ix_0 u}\, \Phi_h(u)du . \tag{3.1}$$

Let F_n be the *modified sample distribution function*; that is, the usual sample distribution function multiplied by $n/(n+1)$. An estimate of $h(x)$ at $x = X_{(j)}$, $h_n(x)$, is

$$h_n(x) = \frac{f_n(x)}{1 - F_n(x)} = \frac{dF_n(x)}{1 - F_n(x)} = \frac{1}{n-j+1} .$$

Let Φ_{h_n} be the Fourier transform of h_n ; that is,

$$\Phi_{h_n}(x) = \int e^{ixu}\, h_n(u)du = \sum_{j=1}^{n} \frac{1}{n-j+1}\, e^{ixX_{(j)}} . \tag{3.2}$$

To obtain from (3.1) an estimate of $h(x_0)$, we replace Φ_h by Φ_{h_n} , and to assure finiteness of the integral, we take it between the finite limits $(-\frac{1}{b(n)}, \frac{1}{b(n)})$, where the $b(n)$ satisfy (2.2), we obtain the *Fourier integral estimator* of $h(x_0)$, $h(n,x_0)$, where

$$\tilde{h}(n,x_0) = \frac{1}{2\pi} \int_{-\frac{1}{b(n)}}^{\frac{1}{b(n)}} e^{-ix_0 u}\, \Phi_{h_n}(u)du . \tag{3.3}$$

Replacing Φ_{h_n} by its expression, (3.2), we have

$$\tilde{h}(n,x_0) = \frac{1}{2\pi}\int_{-\frac{1}{b(n)}}^{\frac{1}{b(n)}} \sum_{j=1}^{n} \frac{1}{n-j+1}\, e^{iu(X_{(j)}-x_0)}\, du$$

$$= \frac{1}{2\pi}\sum_{j=1}^{n}\frac{1}{n-j+1}\int_{-\frac{1}{b(n)}}^{\frac{1}{b(n)}} \{\cos[u(X_{(j)}-x_0)] + i\,\sin[u(X_{(j)}-x_0)]\}du$$

$$= \sum_{j=1}^{n}\frac{1}{n-j+1}\,\frac{\sin\left(\frac{X_{(j)}-x_0}{b(n)}\right)}{\pi(X_{(j)}-x_0)}$$

$$= \sum_{j=1}^{n}\frac{1}{(n-j+1)b(n)}\,\frac{\sin\left(\frac{X_{(j)}-x_0}{b(n)}\right)}{\pi\left(\frac{X_{(j)}-x_0}{b(n)}\right)},$$

or

$$\tilde{h}(n,x_0) = \sum_{j=1}^{n}\frac{1}{(n-j+1)b(n)}\, S\left(\frac{X_{(j)}-x_0}{b(n)}\right), \tag{3.4}$$

where $S(x) = \frac{\sin x}{\pi x}$ is the "sinc" function, illustrated in Figure 3.1.

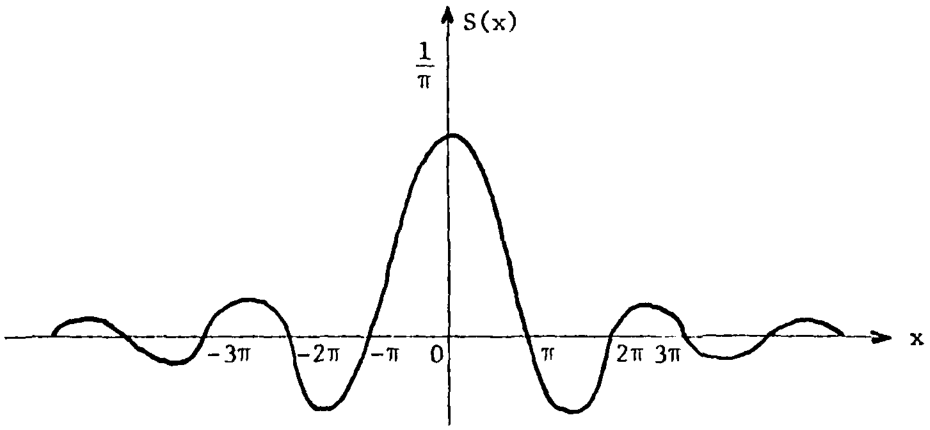

Figure 3.1--The sinc function.

Thus we see that the Fourier integral estimate of the failure rate is indeed a kernel estimate, with the sinc function S as the kernel.

Note that the kernel S is not an L^1 kernel, but that it is symmetric, bounded, and of integral one; also $\int S^2(x)dx = \frac{1}{\pi}$.

4. Asymptotic Unbiasedness and Consistency

Since S is not an L^1 kernel, the asymptotic unbiasedness and consistency of $\tilde{h}(n,x_0)$ has to be established first. Once this is done, we will be able to discuss the rates of convergence of the bias and the MSE.

Theorem 4.1: Let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ be an ordered sample of lifetimes from an absolutely continuous distribution function F . Suppose that:

(i) the failure rate function h is absolutely integrable*;

(ii) h satisfies Dirichlet's conditions in any finite interval; that is, h has at most a finite number of finite discontinuities, and no infinite discontinuities in any finite interval, and, furthermore, h has only a finite number of maxima and minima in any finite interval;

(iii) $h(x)$ is continuous at x_0 ;

(iv) $F(x_0) < 1$; and

---

*When h is absolutely integrable, F is a *subdistribution function* [Chung (1974, p. 84)].

(v) F is such that for any fixed x', and every fixed $\lambda > 0$, there exists a $G_\lambda > 0$, such that

$$\frac{1}{\pi(1-F(x))}\left|\frac{\sin\left(\frac{x-x'}{b(n)}\right)}{x-x'}\right| \leq G_\lambda$$

for all sufficiently large n and for all $|x-x'| \geq \lambda$,

then

$$\tilde{h}(n,x_0) = \sum_{j=1}^{n} \frac{1}{n-j+1} \frac{\sin\left(\frac{X_{(j)}-x_0}{b(n)}\right)}{\pi(X_{(j)}-x_0)}$$

is an asymptotically unbiased and consistent estimator of $h(x_0)$.

Furthermore, an asymptotic expression for the expected value of $\tilde{h}(n,x_0)$ is*

$$E[\tilde{h}(n,x_0)] \sim \int \frac{1}{\pi} \frac{\sin\left(\frac{u-x_0}{b(n)}\right)}{u-x_0} h(u)du , \qquad (4.1)$$

and the variance $\mathrm{Var}[\tilde{h}(n,x_0)]$ converges to zero at the rate $\frac{1}{nb(n)}$.

Proof:

$$E[\tilde{h}(n,x_0)]$$

$$= \sum_{j=1}^{n} \int \frac{1}{n-j+1} \frac{\sin\left(\frac{u-x_0}{b(n)}\right)}{\pi(u-x_0)} f_{X_{(j)}}(u)du$$

$$= \sum_{j=1}^{n} \int \frac{\sin\left(\frac{u-x_0}{b(n)}\right)}{\pi(u-x_0)} \frac{1}{n-j+1} \frac{n!}{(j-1)!(n-j)!} [F(u)]^{j-i}[1-F(u)]^{n-j}f(u)du$$

---

*The notation "$a_n \sim b_n$" denotes the fact that the ratio of a_n to b_n has limit one.

$$= \int \frac{\sin\left(\frac{u-x_0}{b(n)}\right)}{\pi(u-x_0)} \sum_{j=1}^{n} \binom{n}{j-1} (F(u))^{j-1}(1-F(u))^{n-j+1} h(u)du$$

$$= \int \frac{\sin\left(\frac{u-x_0}{b(n)}\right)}{\pi(u-x_0)} h(u)[1 - F^n(u)]du$$

$$= \frac{1}{\pi} \int \frac{\sin\left(\frac{u-x_0}{b(n)}\right)}{u - x_0} h(u)du - \frac{1}{\pi} \int \frac{\sin\left(\frac{u-x_0}{b(n)}\right)}{u - x_0} h(u)F^n(u)du \ . \qquad (4.2)$$

Consider the limit of the first term on the right-hand side of (4.2):

$$\lim_{n\to\infty} \frac{1}{\pi} \int \frac{\sin\left(\frac{u-x_0}{b(n)}\right)}{u - x_0} h(u)du = \lim_{b(n)\to 0} \frac{1}{\pi} \int \frac{\sin\left(\frac{u-x_0}{b(n)}\right)}{u - x_0} h(u)du$$

$$= \lim_{\lambda\to\infty} \frac{1}{\pi} \int \frac{\sin[\lambda(u-x_0)]}{u - x_0} h(u)du$$

$$= h(x_0) \ . \qquad (4.3)$$

The last equality follows by the Fourier integral formula [see Titchmarch (1962, pp. 3, 25) or Smirnov (1964, pp. 462-472)].

Next we show that the second term on the right-hand side of (4.2) tends to zero geometrically, as $n \to \infty$. Since $F(x_0) < 1$, we can choose a $\lambda > 0$ so that $F(x_0+\lambda) < 1$, and such that $h(u)$ is bounded in $|u-x_0| \leq \lambda$. We split the interval of integration $(-\infty,\infty)$ into two parts, $|u-x_0| \leq \lambda$ and $|u-x_0| > \lambda$, and note that

$$\int_{|u-x_0|\leq\lambda} \frac{1}{\pi} \frac{\sin\left(\frac{u-x_0}{b(n)}\right)}{u - x_0} h(u)F^n(u)du \leq (\text{const})F^n(x_0+\lambda) \to 0 \ , \qquad (4.4)$$

as $n \to \infty$, and

$$\int_{|u-x_0|>\lambda} \frac{1}{\pi} \frac{\sin\left(\frac{u-x_0}{b(n)}\right)}{u-x_0} h(u)F^n(u)du = \int_{|u-x_0|>\lambda} \frac{1}{\pi} \frac{\sin\left(\frac{u-x_0}{b(n)}\right)}{u-x_0} \frac{f(u)}{1-F(u)} F^n(u)du$$

$$\leq G_\lambda \int_0^1 F^n dF = \frac{G_\lambda}{n+1} \to 0 \;, \qquad (4.5)$$

as $n \to \infty$.

From (4.2) through (4.5), we conclude that

$$\lim_{n\to\infty} E[\tilde{h}(n,x_0)] = h(x_0) \;.$$

Thus, the estimator $\tilde{h}(n,x_0)$ is asymptotically unbiased and, for a given n ,

$$E[\tilde{h}(n,x_0)] \sim \int \frac{1}{\pi} \frac{\sin\left(\frac{u-x_0}{b(n)}\right)}{u-x_0} h(u)du \;.$$

To prove consistency of $\tilde{h}(n,x_0)$, we follow the detailed steps given in Watson and Leadbetter (1964b). From Equation (3.2) of Watson and Leadbetter, we write

$$\begin{aligned} &\mathrm{Var}[\tilde{h}(n,x_0)] \\ &= \int \frac{1}{b^2(n)} S^2\left(\frac{u-x_0}{b(n)}\right) h(u)\; I_n(F(u))\; dF(u) \\ &+ 2 \int\int_{0\leq u\leq v} \frac{\frac{1}{b(n)} S\left(\frac{u-x_0}{b(n)}\right) \frac{1}{b(n)} S\left(\frac{v-x_0}{b(n)}\right)}{1-F(v)} \left\{\frac{1-F^n(u)}{1-F(u)} F^n(v) \right. \qquad (4.6) \\ &\qquad \left. - \frac{F^n(v)-F^n(u)}{F(v)-F(u)}\right\} dF(u)dF(v) \;, \end{aligned}$$

where $I_n(F) = \int_0^{1-F} \frac{(F+B)^n - F^n}{B} dB$.

If we multiply both sides of (4.6) by n/α_n , where

$$\alpha_n = \int \frac{1}{b^2(n)} S^2\left(\frac{u-x_0}{b(n)}\right) du ,$$

and take the limit as $n \to \infty$, we note that the first term on the right-hand side of (4.6) equals $h(x_0)/(1-F(x_0))$ whereas the second term is 0 . Thus

$$\lim_{n\to\infty} \frac{n}{\alpha_n} \mathrm{Var}[\tilde{h}(n,x_0)] = \frac{h(x_0)}{1-F(x_0)}$$

or that

$$\mathrm{Var}[\tilde{h}(n,x_0)] \sim \frac{\alpha_n}{n} \frac{h(x_0)}{1-F(x_0)} .$$

Since

$$\alpha_n = \int \frac{1}{b^2(n)} S^2\left(\frac{u-x_0}{b(n)}\right) du = \frac{1}{\pi b(n)}$$

implies $\alpha_n/n = (1/\pi)(1/nb(n)) \to 0$, by (2.2), and thus $\mathrm{Var}[\tilde{h}(n,x_0)] \to 0$. Thus, $\tilde{h}(n,x_0)$ is a consistent estimator of $h(x_0)$, and the variance of $\tilde{h}(n,x_0)$ goes to zero at the rate $1/nb(n)$. //.

4.1 An alternate expression for the bias

We shall find it useful to express the asymptotic bias of $\tilde{h}(n,x_0)$ in terms of the Fourier transform Φ_h of h . We first note that if

$$w(x) = \begin{cases} 1 , & |x| \le 1 \\ 0 , & |x| > 1 \end{cases} ,$$

then the inverse Fourier transform of $w(x)$ is

$$\frac{1}{2\pi} \int e^{-ixu} w(u)du = \frac{1}{2\pi} \int_{-1}^{1} e^{-ixu} du = \frac{\sin x}{\pi x} ;$$

That is,

$$S(x) = \frac{1}{2\pi} \int e^{-ixu} w(u)du . \tag{4.7}$$

In view of the above, the Fourier transform of $S(x)$ is $w(x)$; that is

$$\int e^{ixu}\, S(u)du = \int e^{ixu} \frac{\sin u}{\pi u}\, du = \begin{cases} 1 , & |x| < 1 \\ 0 , & |x| > 1 \\ \frac{1}{2} , & |x| = 1 \end{cases} .$$

Recall, from (4.1), that

$$E[\tilde{h}(n,x_0)] \sim \frac{1}{b(n)} \int h(u)\, S\left(\frac{x_0-u}{b(n)}\right) du$$

$$= \frac{1}{b(n)} \int h(u) \left[\frac{1}{2\pi} \int e^{-i\left(\frac{x_0-u}{b(n)}\right)v} w(v)dv\right] du , \quad \text{by (4.7)}$$

$$= \frac{1}{b(n)} \iint h(u)\, \frac{1}{2\pi}\, e^{-i(x_0-u)t}\, w(b(n)t)\, b(n)dt\, du$$

$$= \frac{1}{2\pi} \int e^{-ix_0t}\, w(b(n)t) \int e^{iut}\, h(u)\, du\, dt$$

$$= \frac{1}{2\pi} \int e^{-ix_0t}\, w(b(n)t)\, \Phi_h(t)\, dt .$$

The asymptotic bias of $\tilde{h}(n,x_0)$ is therefore given by

$$\text{Bias}[\tilde{h}(n,x_0)] \sim \frac{1}{2\pi} \int e^{-ix_0t}\, w(tb(n))\, \Phi_h(t)\, dt - h(x_0)$$

$$= \frac{1}{2\pi} \int e^{-ix_0t}\, w(tb(n))\, \Phi_h(t)\, dt - \frac{1}{2\pi} \int e^{-ix_0t}\, \Phi_h(t)\, dt$$

$$= \frac{1}{2\pi} \int e^{-ix_0t}\, \{w(tb(n)) - 1\}\, \Phi_h(t)\, dt$$

$$= -\frac{1}{2\pi} \int_{|t| > \frac{1}{b(n)}} e^{-ix_0t}\, \Phi_h(t)\, dt . \tag{4.8}$$

5. Rates of Convergence of the Bias and the MSE

We are able to investigate and optimize the rates of convergence of the bias and the MSE of $\tilde{h}(n,x_0)$ when the Fourier transform of the failure rate decreases exponentially or algebraically.

Definition 5.1 [Parzen (1958)]: A function $g(x)$ is said to *decrease exponentially* with degree $0 < r \leq 2$, and coefficient $\rho > 0$, if

$$|g(x)| \leq Ae^{-\rho|x|^r} \quad \text{for some constant } A > 0 , \tag{5.1}$$

and

$$\lim_{x\to\infty} \int_0^1 \left[1 + \exp(2\rho x^r)\ |g(xu)|^2\right]^{-1} du = 0 . \tag{5.2}$$

We shall first need to prove the following lemmas.

Lemma 5.1:

$$\lim_{n\to\infty} b(n)\ e^{\frac{1}{b^r(n)}} \int_{\frac{1}{b(n)}}^{\infty} e^{-t^r}\, dt = 0 , \quad (r > 0) . \tag{5.3}$$

Proof: The right-hand side of the above equation, when $\lambda = 1/b(n)$, is

$$\lim_{n\to\infty} \frac{\int_\lambda^\infty e^{-t^r} dt}{\lambda e^{-\lambda^r}} = \lim_{\lambda\to\infty} \frac{-e^{-\lambda^r}}{e^{-\lambda^r} - r\lambda^r e^{-\lambda^r}} = \lim_{\lambda\to\infty} \frac{1}{r\lambda^r - 1} = 0 . \qquad //$$

Lemma 5.2: If the Fourier transform of h , Φ_h , decreases exponentially with degree r and coefficient ρ , then, for sufficiently large n ,

$$|\mathrm{Bias}[\tilde{h}(n,x_0)]| \leq \frac{1}{\pi} \int_{\frac{1}{b(n)}}^{\infty} Ae^{-\rho t^r}\, dt . \tag{5.4}$$

Proof: From (4.8), we note that for n large

$$\mathrm{Bias}[\tilde{h}(n,x_0)] \sim -\frac{1}{2\pi} \int_{|t| > \frac{1}{b(n)}} e^{-ix_0 t}\, \Phi_h(t) dt .$$

However,

$$\left| -\frac{1}{2\pi} \int_{|t| > \frac{1}{b(n)}} e^{-ix_0 t} \Phi_h(t)dt \right| \leq \frac{1}{2\pi} \int_{|t| > \frac{1}{b(n)}} |\Phi_h(t)|dt$$

$$\leq \frac{1}{2\pi} \int_{|t| > \frac{1}{b(n)}} Ae^{-\rho|t|^r} dt = \frac{1}{\pi} \int_{t > \frac{1}{b(n)}} Ae^{-\rho t^r} dt \; ;$$

the statement of the lemma now follows. //.

Lemma 5.3: Suppose that the Fourier transform Φ_h of h decreases exponentially with degree r and coefficient ρ. Then

$$\lim_{n\to\infty} b(n)\, e^{\rho/b^r(n)}\, |\text{Bias}[\tilde{h}(n,x_0)]| = 0 \; ; \tag{5.5}$$

thus

$$\text{Bias}[\tilde{h}(n,x_0)] = o\left(b^{-1}(n)\, e^{-\rho/b^r(n)}\right) .*$$

Proof: The result follows if we make a change of variable $u^r = \rho t^r$, and use Lemmas 5.1 and 5.2. //.

The following theorem establishes the choice of $b(n)$ which enables us to obtain the optimal rate of convergence of the mean square error of $\tilde{h}(n,x_0)$, when Φ_h decreases exponentially.

Theorem 5.4: Suppose that the Fourier transform Φ_h of the unknown failure rate h exists and decreases exponentially with degree $0 < r \leq 2$ and coefficient $\rho > 0$. Then, if $b(n)$ in the Fourier integral estimator of h, $\tilde{h}(n,x_0)$, given by (3.4), is chosen such that $b(n) = O(\log n/2\rho)^{-(1/r)}$, the optimal rate of convergence of the MSE of $\tilde{h}(n,x_0)$ is of the order $\log n/n$.

---

*The notation "$a_n = o(b_n)$" denotes the fact that the ratio of a_n to b_n has limit 0.

Proof: From Lemma 5.3, we note that $\mathrm{Bias}^2[\tilde{h}(n,x_0)]$ decreases at least as fast as $1/b^2(n)\exp(-2\rho/b^r(n))$. From Theorem 4.1, we have the result that the variance of $\tilde{h}(n,x_0)$ goes to zero at the rate $(nb(n))^{-1}$. The statement of the theorem now follows from Davis (1974). //.

We shall now consider the class of failure rate functions h whose Fourier transforms Φ_h decrease algebraically.

Definition 5.2 [Parzen (1958)]: A function $g(x)$ is said to *decrease algebraically* with degree $p > 0$, if

$$\lim_{|x|\to\infty} |x|^p \, |g(x)| = \alpha^{\frac{1}{2}} > 0 , \tag{5.6}$$

for some $\alpha > 0$.

Lemma 5.5: Suppose that the Fourier transform Φ_h of h decreases algebraically with degree $p > 1$. Then

$$\lim_{n\to\infty} b^{1-p}(n) \int_{|t| > \frac{1}{b(n)}} |\Phi_h(t)|dt = 2\alpha^{\frac{1}{2}}(p-1)^{-1} . \tag{5.7}$$

Proof: From (5.6), we note that for $\varepsilon > 0$, there exists an $M > 0$ such that

$$\alpha^{\frac{1}{2}} - \varepsilon < |t|^p \, |\Phi_h(t)| < \alpha^{\frac{1}{2}} + \varepsilon ,$$

whenever $|t| > M$. Equivalently,

$$|t|^{-p}(\alpha^{\frac{1}{2}} - \varepsilon) < |\Phi_h(t)| < |t|^{-p}(\alpha^{\frac{1}{2}} + \varepsilon) ,$$

for $|t| > M$. Integrating both sides of the above for $|t| > 1/b(n)$, we get

$$\begin{aligned} 2(\alpha^{\frac{1}{2}} - \varepsilon)(p-1)^{-1} \, b^{p-1}(n) &< \int_{|t| > \frac{1}{b(n)}} |\Phi_h(t)|dt \\ &< 2(\alpha^{\frac{1}{2}} + \varepsilon)(p-1)^{-1} \, b^{p-1}(n) , \quad |t| > M . \end{aligned}$$

Note that when n is sufficiently large, $1/b(n) > M$, and $|t| > 1/b(n)$ implies that $|t| > M$. Thus, for sufficiently large n,

$$2(\alpha^{\frac{1}{2}} - \varepsilon)(p-1)^{-1} < b^{1-p}(n) \int_{|t| > \frac{1}{b(n)}} |\Phi_h(t)|dt < 2(\alpha^{\frac{1}{2}} + \varepsilon)(p-1)^{-1} .$$

We therefore have

$$\lim_{n\to\infty} \left[b^{1-p}(n) \int_{|t| > \frac{1}{b(n)}} |\Phi_h(t)|dt \right] = 2\alpha^{\frac{1}{2}}(p-1)^{-1} . \qquad //.$$

Lemma 5.6: Suppose that the Fourier transform Φ_h of h decreases algebraically with degree $p > 1$. Then the bias of $\tilde{h}(n,x_0)$, $\text{Bias}[\tilde{h}(n,x_0)]$, satisfies

$$\lim_{n\to\infty} b^{1-p}(n) \; |\text{Bias}[\tilde{h}(n,x_0)]| \leq \alpha^{\frac{1}{2}} \pi^{-1} (p-1)^{-1} ; \tag{5.8}$$

thus the bias decreases at the rate $b^{p-1}(n)$.

Proof: From (4.8), we have

$$b^{1-p}(n) \; |\text{Bias}[\tilde{h}(n,x_0)]| = b^{1-p}(n) \left| \frac{1}{2\pi} \int_{|t| > \frac{1}{b(n)}} e^{-ix_0 t} \Phi_h(t)dt \right|$$

$$\leq b^{1-p}(n) \frac{1}{2\pi} \int_{|t| > \frac{1}{b(n)}} |\Phi_h(t)|dt .$$

Using (5.7), we obtain

$$\lim_{n\to\infty} b^{1-p}(n) \; |\text{Bias}[\tilde{h}(n,x_0)]| \leq \frac{1}{2\pi} 2\alpha^{\frac{1}{2}}(p-1)^{-1} = \alpha^{\frac{1}{2}} \pi^{-1} (p-1)^{-1} . \; //$$

The following theorem is analogous to Theorem 5.4.

Theorem 5.7: Suppose that the Fourier transform Φ_h of the unknown failure rate h exists and decreases algebraically with degree $p > 1$. Then, if $b(n)$ in the Fourier integral estimator of h, $\tilde{h}(n,x_0)$, given by (3.4), is chosen such that $b(n) = O(n^{-1/(2p-1)})$, the optimal rate of convergence of the MSE of $\tilde{h}(n,x_0)$ is of the order $n^{1/(2p-1)-1}$.

Proof: From Lemma 5.6, we note that $\text{Bias}^2[\tilde{h}(n,x_0)]$ decreases at the rate of $b^{2(p-1)}(n)$, and from Theorem 4.1 the variance of $\tilde{h}(n,x_0)$ goes to zero at the rate $(nb(n))^{-1}$. Thus, a choice of $b(n) = O(n^{-1/(2p-1)})$ gives an optimal rate of convergence of MSE of the order $n^{-1+1/(2p-1)}$.
//.

6. A Comparison of the Rates of Convergence of the MSE's

We can now compare the optimal rates of convergence of the MSE's for estimates of h based on L^1 kernels and the sinc function kernel which is not an L^1 kernel.

In general, for L^1 kernels which belong to the class A_m (i.e., an L^1 kernel K which satisfies the condition that $\int x^j K(x)dx = 0$, for $j=1,2,\ldots,m-1$), and if $h^{(m)}$ exists (that is, if h is m times continuously differentiable), then we have shown in SW(1980) that the optimal rate of convergence of the MSE of the kernel estimator of h is of the order $n^{-2m/(2m+1)}$.

In Theorem 5.4 of this paper, we have shown that if the Fourier transform of h, Φ_h, decreases exponentially with degree r and coefficient ρ, then the optimal rate of convergence of the MSE of the Fourier integral estimator of h is of the order $\log n/n$. Since

$$\lim_{n\to\infty} \frac{\log n/n}{n^{-2m/(2m+1)}} = \lim_{n\to\infty} \frac{\log n}{n^{-1/(2m+1)}} = 0 ,$$

for all $m > 0$, the following theorem follows.

Theorem 6.1: For the class of failure rate functions whose Fourier transform exists, and decreases exponentially, the Fourier integral estimate (based on the sinc function) is better in terms of the rate of convergence of the MSE than a kernel estimate based on any L^1 kernel.

When the Fourier transform of h , Φ_h , decreases algebraically, then in Theorem 5.7 we have shown that the optimal rate of convergence of the MSE of the Fourier integral estimator of h is of the order $n^{1/(2p-1)-1}$. Since

$$\lim_{n\to\infty} \frac{n^{1/(2p-1)-1}}{n^{-2m/(2m+1)}} = \lim_{n\to\infty} n^{1/(2p-1)-1/(2m+1)} = 0 ,$$

if $p > m+1$, the following theorem holds.

Theorem 6.2: For the class of failure rate functions whose Fourier transform exists and decreases algebraically with degree p , the Fourier integral estimate (based on the sinc function) is better in terms of the MSE than a kernal estimate based on any L^1 kernel belonging to the class A_m , if $p > m+1$.

6.1 Examples of Fourier transforms which decrease exponentially and algebraically

(i) Suppose that the failure rate function h is of the form

$$h(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2t}} t^{-3/2} , \quad t > 0 .$$

The Fourier transform of h , Φ_h , is

$$\Phi_h(t) = \exp\{-|t|^{\frac{1}{2}} (1 + i \frac{t}{|t|})\} .$$

Since $|\Phi_h(t)| = e^{-|t|^{\frac{1}{2}}}$, Φ_h decreases exponentially with $A = 1$, $\rho = 1$, and $r = \frac{1}{2}$. By Theorem 6.1, for this h the Fourier integral estimate is better than kernel estimates based on L^1 kernels.

(ii) Suppose that the failure rate function h is of the form

$$h(t) = \frac{1}{2^{\gamma/2}\Gamma(\gamma/2)} t^{\frac{\gamma}{2}-1} e^{-\frac{\gamma}{2}} ,$$

$t > 0$, γ a positive integer. For $\gamma \geqq 2$, this failure rate function first increases and then decreases. The Fourier transform of h , Φ_h , is

$$\Phi_h(t) = (1 - 2it)^{-\gamma/2} ,$$

and

$$|\Phi_h(t)| = \left(1 + 4t^2\right)^{-\gamma/4} ,$$

implying that Φ_h decreases algebraically with degree $\gamma/2$. By Theorem 6.2, for this h the Fourier integral estimate of h is better than kernel estimates based on L^1 kernels which belong to A_m , whenever $\gamma > 2(m+1)$.

(iii) Suppose that h is of the form

$$h(t) = \lambda e^{-\lambda t} , \quad t > 0 , \lambda > 0 ;$$

that is, the failure rate is an exponential function. The Fourier transform of h , Φ_h , is

$$\Phi_h(t) = \frac{\lambda}{\lambda - it} ,$$

and

$$|\Phi_h(t)| = \left(1 + \frac{t^2}{\lambda^2}\right)^{-\frac{1}{2}} ,$$

implying that Φ_h decreases algebraically with degree 1. By Theorem 6.1, for this h the Fourier integral estimate is inferior to any kernel estimator based on an L^1 kernel.

7. Concluding Remarks

It is evident from Theorems 5.4 and 5.7 that one would consider using the Fourier integral estimate of h only when one had some prior knowledge about the general behavior of h. For example, if we know a priori that the failure rate is an exponential function, then we would use an L^1 kernel rather than a sinc kernel to estimate h. In real life, this kind of prior knowledge may be hard to come by; after all, the practical problem is to estimate an unknown h. This, therefore, poses a disadvantage of the sinc kernel estimators of h.

Another disadvantage of the sinc kernel estimator stems from the fact that the sinc function takes negative value for a substantial number of points in its domain. Thus, the sinc function estimator of h can be negative at some points, a result which is unacceptable to a practitioner. One may argue that this is the price that must be paid for obtaining an estimator which has good bias and MSE properties. On the other hand, a Bayesian may view this as another situation wherein unbiased estimation and MSE minimization lead us to unacceptable answers. Efron (1978), in a delightfully written paper, discusses some controversies in the foundations of statistics.

Our final remark pertains to an interpretation as to why the MSE of the sinc kernel estimator of h has a faster rate of convergence than those obtained via other kernels. In SW(1980), we observe that an indefinite jackknifing of L^1 kernel estimators of h is equivalent to obtaining a kernel estimator of h using an alternating kernel that is not L^1. The sinc kernel, being not L^1 and wavelike, can be viewed as one that could be the consequence of an indefinite amount of jackknifing. Since the result of jackknifing is a reduction of the bias and an improvement of the rate of convergence of the MSE [Miller (1978)], the fast rates of convergence of the MSE follow.

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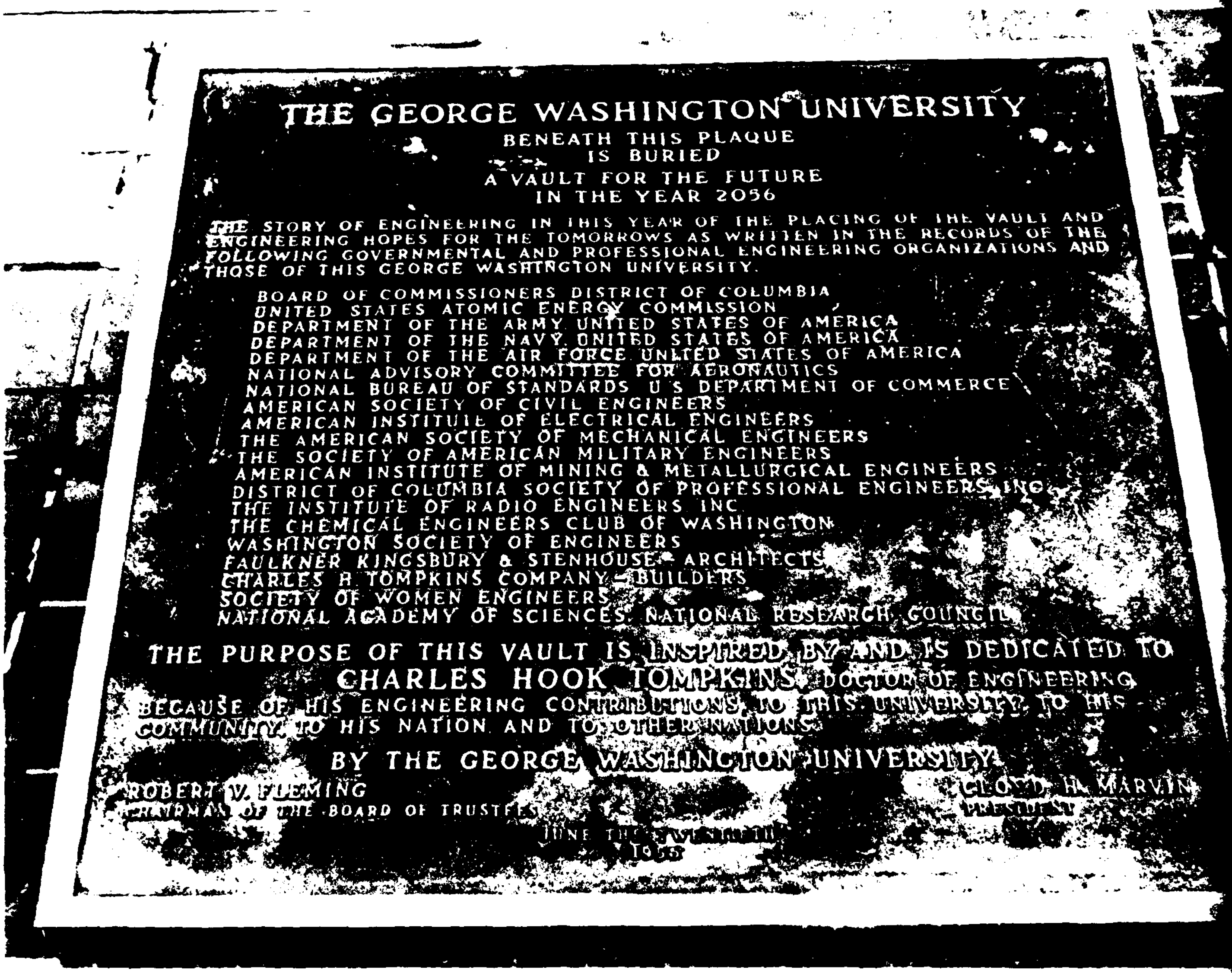

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